A NOTE ON GENERA

Abstract. This note is to be considered expository. We present a topologically motivated survey of the Witten genus, which includes an attempt to overcome a common looseness (or inaccuracy) in the scattered literature on genera or more specifically on the Witten genus. This note is by no means complete, but work in progress.

Contents

1. Graded rings

We start with the definition of a graded ring, which for topologists should slightly differ from the definition of a graded ring that for example an algebraic geometrist uses. This is because of certain subtleties which matter in topology, but are just not visible in algebraic geometry.

Definition 1.1. A *graded ring* R_* is a collection $\{M_i\}_{i\in\mathbb{Z}}$ of modules M_i, called the *homogeneous pieces* of R_*, together with a (graded) multiplication

$$M_i \times M_j \longrightarrow M_{i+j}.$$

Elements of M_i are said to be of degree i. The addition in R_* is the usual addition of sequences.

Notation 1.2. We will denote our graded rings ambiguously by

$$R_* = \bigoplus_{i\in\mathbb{Z}} M_i \qquad \text{and} \qquad R_* = \prod_{i\in\mathbb{Z}} M_i.$$

Both are common notations for graded rings, although the first is even more common than the second. Note that in the context of graded rings the objects representing both notations are in fact *isomorphic*, since in both cases the collection of homogeneous pieces is the same. This is of course false in the category of rings, as one quickly recognizes when comparing the underlying sets.

As an explanation for our definition we offer the following reason.

Remark 1.3. The algebraic geometry definition of a graded ring, being the direct *sum* of some collection of modules, is, as noted above, slightly inconvenient for topologists. One example of such inconvenience being the following. Cohomology theories should take value in the category of graded rings. On the other hand, a cohomology ring is the direct *product* of all cohomology groups.

While this discrepancy is not visible as long as one only deals with, say, finite complexes, one actually runs into subtleties, once one wants to calculate the cohomology of an infinite complex such as $\mathbb{C}P^\infty$ (cf. Appendix A). Now the direct sum and the direct product will no longer coincide and one has to decide whether the cohomology of $\mathbb{C}P^\infty$ should be a polynomial ring or a power series ring.

There is a natural way to go from the category of graded rings to the category of rings. Consider the forgetful functor

$$\{graded\ Rings\} \overset{forgetful}{\longrightarrow} \{Rings\}$$

which sends a graded ring R_* to the direct product $\prod_{i\in\mathbb{Z}} M_i$ of its homogeneous pieces. This is sometimes referred to as the (*honest*) ring *underlying* R_*.

Remark 1.4. We hope that our reasons for understanding genera as homomorphisms of graded rings becomes more clear along the following sections. In these sections we will talk about (topological) morphisms of ring spectra, which on homotopy groups reproduce certain genera. In this context it is only natural to consider genera as homomorphisms of graded rings, as the homotopy groups of spectra naturally have this structure.

2. Genera

In the common literature such as e.g. [HBJ92] a genus is described as a ring homomorphism

$$\varphi\colon \Omega_* \longrightarrow R$$

from a cobordism ring Ω_* into some ring R. This deserves a to be made a little more precise. The domain of this ring homomorphism is obviously a graded ring, whereas the codomain can be any ring. The natural way to make sense of the term 'ring homomorphism' is to consider the (honest) ring underlying the cobordism ring.

The upshot of this section is that we want to emphasize is that this definition throws away a certain piece of information which might have been superfluous back in the days this definition was proposed, but can turn out be useful in modern topology.

Let us offer a slightly modified version of this definition.

Definition 2.1. A *genus* is a degree preserving homomorphism of graded rings

$$\varphi_*\colon \Omega_*^G \longrightarrow R_*$$

from some cobordism ring, classifying cobordism classes of compact manifolds with G-structure on their tangent bundle, to some graded ring R_*.

A very good question is 'How do the established genera fit into this context?'. The easy answer is, by turning the codomain R into a graded ring R_*. We do so in the most naive way

$$\varphi_*\colon \Omega_*^G \longrightarrow R_* := R[u^{\pm}],$$

where the formal variable u is given some (suitable) degree and the map is required to be degree preserving.

Employing the canonical isomorphisms $R \cong R \cdot u^i$ for all $i \in \mathbb{Z}$ one can also win back the 'original' morphism by considering the following composition

$$\varphi\colon \Omega_*^G \longrightarrow R_* = \begin{pmatrix} \vdots \\ R \\ R \\ R \\ R \\ \vdots \end{pmatrix} \xrightarrow{+} R.$$

This should be considered to "undo" turning R into a graded ring by forgetting the grading and identifying all copies of R.

We hope that the following sections shed some light on the usefulness of this point of view.

Rational genera. If the codomain R_* does not contain additive torsion (i.e. none of the M_i has torsion), then φ_* is of course determined by its rationalization

$$\varphi_*^{\mathbb{Q}}\colon \Omega_*^G \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow R_*.$$

Even if the codomain contains torsion, much of the information of φ is captured by its rationalization. In particular, given a rational genus

$$\varphi_*^{\mathbb{Q}}\colon \Omega_*^G \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow R_* \otimes_{\mathbb{Z}} \mathbb{Q}$$

as usual we have

- a logarithm
$$\log_\varphi(x) := \sum_{i>0} \frac{\varphi(\mathbb{C}P^{2i})}{i+1} \cdot x^{i+1}$$
- a Hirzebruch characteristic series
$$Q_\varphi(x) = \frac{x}{\exp_\varphi(x)}$$
with $\exp_\varphi(x)$ being the exponential of the genus and the inverse of the logarithm $\log_\varphi(x)$.
- a characteristic class
$$Q_\varphi(c_1(L_{univ})) \in H^2(BU(1), R_0 \otimes \mathbb{Q})$$
given by evaluating the Hirzebruch characteristic series on the first Chern class of the universal bundle over $BU(1)$.

  This characteristic class gives rise to other characteristic classes, e.g. for the sum $L_1 \oplus \cdots \oplus L_k$ of line bundles (note that by the splitting principle any bundle has such presentation)
$$\prod_i Q_\varphi(c_1(L_i)).$$

One can thus easily evaluate rational genera on a (cobordism class of a) compact, differentiable, oriented $4n$-dim manifold via

$$\varphi_*(M) = \left\langle \prod_{i=1}^n Q_\varphi(c_1(L_i)), [M] \right\rangle,$$

i.e. evaluating the n-fold product of Q on the fundamental class $[M]$ of M. The L_i are the summands of the decomposition of the complexified tangent bundle $TM \otimes \mathbb{C}$ via the splitting principle. The evaluation of the product is the same as the evaluation on the highest cohomology class, since all other pair to zero.

Remark 2.2. Note that, in order to keep notation simple, we suppress the $*$-index of φ_*, whenever φ_* itself occurs as index.

3. Modular forms

Let $\mathcal{M}_{ell}$ be the moduli stack of elliptic curves and ω be the invertible sheaf on $\mathcal{M}_{ell}$ locally given by the invertible sheaf of differential 1-forms $p_*\Omega^1_E$ for an elliptic curve $p\colon E \longrightarrow S$ and some base scheme S.

One usually denotes by

$$MF^{alg}_* := H^0(\mathcal{M}_{ell}, \omega^{\otimes *}) := \bigoplus_k H^0(\mathcal{M}_{ell}, \omega^{\otimes k})$$

the (graded) ring of modular forms. In the relevant literature the graded ring MF^{alg}_* is usually denoted by just MF_*. We choose to decorate this ring with the further index *alg* in order to emphasize that the grading is algebraic in the sense that the weight k is an algebraic one.

It is well known that there are only modular forms of even weight, i.e.

$$MF^{alg}_{2k+1} = H^0(\mathcal{M}_{ell}, \omega^{\otimes 2k+1}) = 0.$$

Another well known fact is that

$$MF^{alg}_* \cong \mathbb{Z}[E_4, E_6, \Delta]/(E_4^3 - E_6^2 - 1728\Delta)$$

where the E_{2k} are Eisenstein series of weight $2k$, normalized such that their q-expansion have only integral coefficients and starts with 1. For some arithmetic group $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ these Eisenstein series are defined by

$$\begin{aligned} E_{2k}(\tau) &= \sum_{\gamma \in (P\cap\Gamma)\backslash\Gamma} 1_{2k}\big|\gamma \\ &= \frac{1}{2} \sum_{\substack{(c,d)=1 \\ c,d\in\mathbb{Z}}} \frac{1}{(c\tau+d)^{2k}} \\ &= 1 + \frac{4k}{-B_{2k}} \sum_{n\geq 1} \sigma_{2k-1}(n) q^n. \end{aligned}$$

Here P is the minimal/maximal parabolic subgroup of the full modular group $\mathrm{SL}_2(\mathbb{Z})$, as usual (c,d) denotes the *gcd* of c and d and $\sigma_m(n)$ is the sum $\sum_{d|n} d^m$ of m-th powers of the positive divisors d of n. The notation $f_n\big|\gamma$ denotes

$$f_n\big|\gamma(x) = j(x,\gamma)^{-n} f(\gamma.x)$$

where j denotes the factor of automorphy for Γ.

Considering a different normalization

$$G_{2k}(\tau) = \frac{-B_{2k}}{4k} \cdot E_{2k}$$

of those Eisenstein series, one realizes that the coefficients in their q-expansions

$$G_{2k}(\tau) = \frac{-B_{2k}}{2k} + \sum_{i \geq 1} \sigma_{2k-1}(i) q^i$$

remain integral except for the constant one (which is still rational though). In this normalization the Eisenstein series have the further property that they are normalized eigenfunctions for Hecke operators.

Algebraic weight vs. topological weight. As mentioned above the ring of modular forms is commonly graded via the algebraic weight of modular forms. In this section we would like to provide arguments for the introduction of a different, i.e. topological, grading. In short we argue that the invertible sheaf ω itself deserves to be of degree 2 and hence define

$$MF^{top}_{2*} := MF^{alg}_{*}.$$

This grading will be much more suitable to our perspective that genera are degree preserving homomorphism of graded rings.

The construction of the ring spectrum tmf (or its periodic versions) of topological modular forms is a consequence of the derived structure of the moduli stack of elliptic curves, hence depends crucially on the construction of a sheaf, usually denoted by $\mathcal{O}^{top}$, of E_∞-ring spectra on $\mathcal{M}_{ell}$. Evaluated on étale affines $S = \mathrm{Spec}(R)$ over $\mathcal{M}_{ell}$ the sheaf $\mathcal{O}^{top}$ produces weakly even periodic E_∞-ring spectra. These are E_∞-ring spectra E such that $\pi_* E$ is concentrated in even degrees, i.e. $\pi_{2k+1}E = 0$, $\pi_2 E$ is an invertible (i.e. locally free of rank one) $\pi_0 E$-module and the multiplication maps

$$\pi_2 E \otimes_{\pi_0 E} \pi_{2t} E \longrightarrow \pi_{2t+2} E$$

are isomorphisms. Note that locally the concept of weakly even periodic and even periodic ($\pi_2 E$ is a free $\pi_0 E$-module of rank one) of course coincide.

Let E be a weakly even periodic ring spectrum given by evaluating $\mathcal{O}^{top}$ on some étale open S, then the E-cohomology of $\mathbb{C}P^\infty$ is described in Appendix A. Note that $\mathrm{Spf}(E^0(\mathbb{C}P^\infty))$ is by construction of $\mathcal{O}^{top}$ isomorphic to the formal group $\mathbb{G}_E \longrightarrow S$ associated to the cohomology represented by E. The ring of functions of $\mathbb{G}_E$ is thus given by $E^0(\mathbb{C}P^\infty)$. Let us denote by I the ideal sheaf on S defined by those functions on $\mathbb{G}_E$ vanishing of on the identity section. A common argument shows that I/I^2 is (isomorphic to) the (Zariski) cotangent space/sheaf (on S) of $\mathbb{G}_E$ at the identity section. On the other hand the natural maps

$$I/I^2 \longrightarrow E^{-2} \cdot x \xrightarrow[can.]{\cong} E^{-2} \cong \pi_2(E)$$

are isomorphisms.

An easy argument shows that the cotangent spaces of the elliptic curve C_S and the cotangent space/sheaf of the formal group $\widehat{C} \cong \mathbb{G}_E$ are isomorphic, in fact coincide. Since the cotangent sheaf of C_S is the sheaf of differential 1-forms and we have ω defined to be the latter, we have thus shown that locally (i.e. on étale opens$S$) the invertible sheaf ω is isomorphic to $\pi_2(E)$.

This argument is the reason to consider ω to be of topological degree 2 and is the justification for the definition of

$$MF^{top}_{2*} := MF^{alg}_{*} := H^0(\mathcal{M}_{ell}, \omega^{\otimes *}).$$

Since there are only modular forms of even algebraic weight, the topologically graded ring of modular forms is concentrated in degrees, which are a multiple of 4, i.e.

$$MF^{top}_{4k} = MF^{alg}_{2k} = H^0(\mathcal{M}_{ell}, \omega^{\otimes 2k}).$$

It is still correct, that

$$MF^{top}_* \cong \mathbb{Z}[E_4, E_6, \Delta]/(E_4^3 - E_6^2 - 1728\Delta)$$

only the degrees of E_4, E_6 and Δ are $8, 12$ and 24 now. Note that these generators are topologically realized by maps

$$S^8 \longrightarrow tmf$$
$$S^{12} \longrightarrow tmf$$

and

$$S^{24} \longrightarrow tmf.$$

4. The Witten genus

There is a famous Hirzebruch characteristic series

$$Q_W(x,\tau) = exp\left(\sum_{k\geq 1} \frac{2G_{2k}(\tau)}{(2k)!} \cdot x^{2k}\right),$$

representing a rational genus, that when evaluated as

$$\left\langle \prod_i Q_W(x_i), [M] \right\rangle \in \mathbb{Q}[[x]]$$

on a compact oriented manifold M of dimension $4k$ gives a rational power series.

This rational genus is called the (rational) *Witten genus*,

$$\varphi^{\mathbb{Q}}_W : MSO^{\mathbb{Q}}_* \longrightarrow \bigoplus_{4k} \mathbb{Q}[[q]].$$

Since rationalization is a ring homomorphism, we obtain a degree preserving homomorphism

$$\begin{tikzcd} MSO_* \arrow[d] \arrow[dr, "\varphi_W"] & \\ MSO^{\mathbb{Q}}_* \arrow[r, "\varphi^{\mathbb{Q}}_W"] & \bigoplus_{4k} \mathbb{Q}[[q]]. \end{tikzcd}$$

of graded rings. Since the bottom horizontal map is a (degree preserving) homomorphism of graded $\mathbb{Q}$-algebras, the composition homomorphism will send all torsion in MSO_* to zero.

An important observation is that considering only the constant term $\frac{-B_{2k}}{4k}$ of the q-expansion of the Eisenstein series instead of the Eisenstein series G_{2k} themselves yields the characteristic series of the $\widehat{A}$-genus.

Thus by integrality of the $\widehat{A}$-genus on *Spin*-manifolds and the integrality of the (non-constant term) coefficients of the G_{2k} we easily see that φ_W takes values in integral power series when evaluated on *Spin*-manifolds.

$$\begin{tikzcd}
MSpin_* \arrow[d] \arrow[dr, "\varphi_W"] & \\
MSO_* \arrow[d] \arrow[dr, "\varphi_W"] & \bigoplus_{4k} \mathbb{Z}[[q]] \arrow[d, hook] \\
MSO_*^{\mathbb{Q}} \arrow[r, "\varphi_W^{\mathbb{Q}}"] & \bigoplus_{4k} \mathbb{Q}[[q]].
\end{tikzcd}$$

We slightly abuse notation by denoting the upper genus by φ_W as well. We will also call this genus *Witten genus*. Let us also remark, that as before this morphism will send all torsion to zero.

The way the characteristic series is given one would expect that the genus would take values in another subring of $\bigoplus_{4k} \mathbb{Z}[[q]]$ namely in the (graded) ring MF_* of modular forms. Unfortunately, the Eisenstein series G_2 is not a modular form, but only a quasi-modular form (i.e. G_2 does not transform quite as modular as expected by a modular form).

Analyzing the evaluation of characteristic classes given by the characteristic series $Q_\varphi(x)$ on the fundamental class of the $4n$-dimensional manifold one realizes that the quasi-modular form G_2 is tied to the square x_i^2 of the Chern roots of the tangent bundle, or more precisely to the sum $\sum_{i=1}^{2n} x_i^2$ of those (as one takes the product of the $Q_W(x_i)$ over all i).

By definition this is the first Pontryagin class

$$p_1(M) = \sum_{i=1}^{2n} x_i^2$$

of (the tangent bundle of) the manifold M. Thus if p_1 vanishes G_2 would not appear in the evaluation and thus the genus φ_W would actually take values in MF_*^{alg}.

Note that p_1 is twice the generator of $H^4(BSpin, \mathbb{Z})$, which as a consequence is commonly denoted by $\frac{p_1}{2}$. The structure on *Spin*-manifolds requiring $\frac{p_1}{2}$ being 2-torsion is called a *rational String*-structure. Considering only such manifolds hence yields

$$\begin{tikzcd}
\Omega_*^{\mathbb{Q}-String} \arrow[d] \arrow[dr, dashed, "\varphi_W"] & \\
MSpin_* \arrow[d] \arrow[dr, "\varphi_W"] & MF_*^{alg} \arrow[d, hook, dashed] \\
MSO_* \arrow[d] \arrow[dr, "\varphi_W"] & \bigoplus_{4k} \mathbb{Z}[[q]] \arrow[d, hook] \\
MSO_*^{\mathbb{Q}} \arrow[r, "\varphi_W^{\mathbb{Q}}"] & \bigoplus_{4k} \mathbb{Q}[[q]].
\end{tikzcd}$$

There are two defects in considering MF_*^{alg} in this context. We cannot identify MF_*^{alg} as graded subring of $\bigoplus_{4k} \mathbb{Z}[[q]$, i.e. the the dashed vertical arrow does not exist. The second defect is that the genus φ_W cannot be a degree preserving homomorphism of graded rings, since a $4k$-dimensional manifold gets send to a modular form of algebraic weight $2k$. Both defects are strong indicators that we should use the ring MF_*^{top} instead. In fact,

$$\begin{tikzcd}
\Omega_*^{\mathbb{Q}-String} \arrow[d] \arrow[dr, "\varphi_W"] & \\
MSpin_* \arrow[d] \arrow[dr, "\varphi_W"] & MF_*^{top} \arrow[d, hook] \\
MSO_* \arrow[d] \arrow[dr, "\varphi_W"] & \bigoplus_{4k} \mathbb{Z}[[q]] \arrow[d, hook] \\
MSO_*^{\mathbb{Q}} \arrow[r, "\varphi_W^{\mathbb{Q}}"] & \bigoplus_{4k} \mathbb{Q}[[q]].
\end{tikzcd}$$

commutes in the category of graded rings with degree preserving homomorphisms. (Note that some people tend to denote $\Omega_*^{\mathbb{Q}-String}$ by $MStrong_*$.)

By work of [AHR10] we know that if we consider manifolds with actual *String*-structures, i.e. the vanishing of $\frac{p_1}{2}$, then the genus φ_W lifts to the homotopy groups of tmf.

$$\begin{tikzcd}
MString_* \arrow[d] \arrow[dr, "\varphi_W"] & \\
\Omega_*^{\mathbb{Q}-String} \arrow[d] \arrow[dr, "\varphi_W"] & \pi_* tmf \arrow[d, "edge\ hom"] \\
MSpin_* \arrow[d] \arrow[dr, "\varphi_W"] & MF_*^{top} \arrow[d, hook] \\
MSO_* \arrow[d] \arrow[dr, "\varphi_W"] & \bigoplus_{4k} \mathbb{Z}[[q]] \arrow[d, hook] \\
MSO_*^{\mathbb{Q}} \arrow[r, "\varphi_W^{\mathbb{Q}}"] & \bigoplus_{4k} \mathbb{Q}[[q]].
\end{tikzcd}$$

Note that the map form the (graded ring of) homotopy groups of tmf to the graded ring of modular forms is the edge homomorphism in the descent spectral sequence calculating the homotopy groups of tmf.

At this level something new happens in that the Witten genus pics up torsion information for the first time. More precisely, the homotopy groups of tmf contain torsion at the primes 2 and 3, cf. [Bau08], and the map φ_W is (at least conjecturally) surjective and thus pics up torsion information of $MString_*$. The diagram is still commutative since the edge homomorphism forgets about all torsion.

It is indispensable to mention that the authors of [AHR10] show much more than what we mentioned so far. In fact, they produce a map of E_∞-ring spectra

$$MString \longrightarrow tmf$$

which realizes the Witten genus φ_W on homotopy groups.

.... say sth about Hill-Lawson topological q-expansion map from tmf to KO((q)).

5. Complex vs. real/ rational genus

Note that the Witten genus is a priori a complex genus, that by work of [AHS01] lifts to a map

$$MU\langle 6\rangle \longrightarrow K((q))$$

from the Thom spectrum associated to the 6-th connected cover of BU to Tate K-theory spectrum. The fact that its characteristic series contains at least square powers of Chern roots, i.e. Pontryagin roots, turns it into a real genus and hence

$$\begin{tikzcd} MString \arrow[dr] & MU\langle 6\rangle \arrow[l] \arrow[d] \\ & K((q)) \end{tikzcd}$$

above factorization over $MString$. It turns out that rationally, real Tate K-theory, i.e. Tate KO-theory $KO((q))$, is much closer to $\oplus_{4k}\mathbb{Q}[[q]]$, than (complex) Tate K-theory $KU((q))$ which has sort of too much rational information. In fact, it turns out that the Witten genus also factors over $KO((q))$,

$$\begin{tikzcd} MString \arrow[d] & MU\langle 6\rangle \arrow[l] \arrow[d] \\ KO((q)) \arrow[r, "\otimes\mathbb{C}"] & K((q)). \end{tikzcd}$$

Much of the information about the a priori complex Witten genus is contained in the (real/ rational) genus

$$MString \longrightarrow KO((q)).$$

Now according to our above discussion and by work of Ando, Hopkins and Rezk, this map factors over tmf

$$\begin{tikzcd} MString \arrow[d] & MU\langle 6\rangle \arrow[l] \arrow[dd] \\ tmf \arrow[d] & \\ KO((q)) \arrow[r, "\otimes\mathbb{C}"] & K((q)). \end{tikzcd}$$

Combining this with our diagram from above yields the totally commutative diagram

$$
\begin{tikzcd}
MString_* \arrow[d] \arrow[dr, "\varphi_W"] & & & MU\langle 6\rangle_* \arrow[lll] \arrow[ddd] \\
\Omega_*^{\mathbb{Q}-String} \arrow[d] \arrow[dr, "\varphi_W"] & \pi_* tmf \arrow[d, "edge"] \arrow[dr] & & \\
MSpin_* \arrow[d] \arrow[dr, "\varphi_W"] & MF_*^{top} \arrow[d, hook] & \pi_* KO[[q]] \arrow[d, "edge"] \arrow[dr, "(\otimes\mathbb{C})_*"] & \\
MSO_* \arrow[d, "- \otimes \mathbb{Q}"'] \arrow[dr, "\varphi_W"] & \bigoplus_{4k} \mathbb{Z}[[q]] \arrow[r, hook] \arrow[d, hook, "- \otimes \mathbb{Q}"] & R_* \arrow[r] \arrow[d, "- \otimes \mathbb{Q}"] & \bigoplus_{2k} \mathbb{Z}[[q]] = K((q))_* \arrow[d, hook] \\
MSO_*^{\mathbb{Q}} \arrow[r, "\varphi_W^{\mathbb{Q}}"] & \bigoplus_{4k} \mathbb{Q}[[q]] \arrow[r, hook] & \bigoplus_{4k} \mathbb{Q}[[q]] \arrow[r, hook] & \bigoplus_{2k} \mathbb{Q}[[q]] = K((q))_*^{\mathbb{Q}}
\end{tikzcd}
$$

where R_* is the usual graded ring describing the (eight periodic) homotopy of KO adjoint a Laurent series variable q

$$R_* = \begin{cases} \mathbb{Z}((q)) & \text{if } * \equiv 0 \ (4) \\ \mathbb{Z}/2((q)) & \text{if } * \equiv 1, 2 \ (8) \end{cases}$$

6. To be written: Thoughts on TAF related genera

This section has yet to be written. We want to further enlarge the above diagram in any possible way to the level of TAF-spectra. This includes summarizing our work on the analog of the map

$$tmf \longrightarrow K((q))$$

in the TAF-setting, i.e.

$$TAF_n \longrightarrow TAF_{n-1}((s_1))$$

and its iterations

$$TAF_n \longrightarrow TAF_{n-1}((s_1)) \longrightarrow \cdots \longrightarrow TAF_1((s_1, \dots, s_{n-1})).$$

Maybe something on

$$TAF_n \longrightarrow TAF_{n-1}((s_1)) \longrightarrow \cdots \longrightarrow TAF_3((s_1, \dots, s_{n-3})) \longrightarrow tmf_p((s_1, \dots, s_{n-3}, s)).$$

Further this section should include some thoughts on orientations of TAF, at least on the level of homotopy. We mainly think about this in terms of characteristic series.

$$
\begin{tikzcd}
M???_* \arrow[d] \arrow[dr, dashed] & & \\
MString_* \arrow[r, dashed] \arrow[dr, dashed] & \pi_* TAF_n \arrow[d, "\substack{edge \\ hom}"] \arrow[dr] & \\
& (AF_n)_*^{top} \arrow[dr, hook, "exp"] & \pi_* TAF_{n-1}((s_1)) \arrow[d, "\substack{edge \\ hom}"] \\
& & (AF_{n-1})_*^{top}[[s_1]]
\end{tikzcd}
$$

Note that the power series variable s_1 of the graded ring $(AF_{n-1})_*^{top}[[s_1]]$ should be given degree -1. Thus the graded ring has the form

$$(AF_n)_*^{top}[[s_1]] = \begin{cases} \quad\cdots & \\ (AF_n)_4 \times (AF_n)_5 \cdot s_1 \times (AF_n)_6 \cdot s_1^2 \times (AF_n)_7 \cdot s_1^3 \times \cdots & \text{deg 4} \\ (AF_n)_0 \times (AF_n)_1 \cdot s_1 \times (AF_n)_2 \cdot s_1^2 \times (AF_n)_3 \cdot s_1^3 \times \cdots & \text{deg 0} \\ (AF_n)_{-4} \times (AF_n)_{-3} \cdot s_1 \times (AF_n)_{-2} \cdot s_1^2 \times (AF_n)_{-1} \cdot s_1^3 \times \cdots & \text{deg -4} \\ \quad\cdots & \end{cases}$$

Note that these degrees are topological degrees, as indicated by the index "*top*", algebraic degrees will as before be half the topological ones.

Appendix A. The cohomology of $\mathbb{C}P^\infty$

Let E be a weakly periodic cohomology theory. The cohomology of $\mathbb{C}P^\infty$ is the following graded ring

$$\begin{aligned} E^*(\mathbb{C}P^\infty) &\cong E^*[[x]] \\ &= \prod_k \prod_{i+j=k} E^i \cdot x^j \\ &= \begin{cases} \quad\cdots & \\ \prod_{i+j=2} E^i \cdot x^j & \text{deg 2} \\ \prod_{i+j=0} E^i \cdot x^j & \text{deg 0} \\ \quad\cdots & \end{cases} \\ &= \begin{cases} \quad\cdots & \\ E^2 \times E^0 \cdot x \times E^{-2} \cdot x^2 \times E^{-4} \cdot x^3 \times \cdots & \text{deg 2} \\ E^0 \times E^{-2} \cdot x \times E^{-4} \cdot x^2 \times E^{-6} \cdot x^3 \times \cdots & \text{deg 0} \\ E^{-2} \times E^{-4} \cdot x \times E^{-6} \cdot x^2 \times E^{-8} \cdot x^3 \times \cdots & \text{deg -2} \\ \quad\cdots & \end{cases} \end{aligned}$$

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